

Interactions of Charmed Mesons with Light Pseudoscalar Mesons from Lattice QCD and Implications on the Nature of the $D_{s0}^*(2317)$

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We study the scattering of light pseudoscalar mesons (π , K) off charmed mesons (D , D_s) in full lattice QCD. The S -wave scattering lengths are calculated using Lüscher's finite volume technique. We use a relativistic formulation for the charm quark. For the light quark, we use domain-wall fermions in the valence sector and improved Kogut-Susskind sea quarks. We calculate the scattering lengths of isospin-3/2 $D\pi$, $D_s\pi$, D_sK , isospin-0 $D\bar{K}$ and isospin-1 $D\bar{K}$ channels on the lattice. For the chiral extrapolation, we use a chiral unitary approach to next-to-leading order, which at the same time allows us to give predictions for other channels. It turns out that the interpretation of the $D_{s0}^*(2317)$ as a DK molecule is consistent with the results. At the same time, we also update a prediction for the isospin breaking hadronic decay width $\Gamma(D_{s0}^*(2317) \rightarrow D_s\pi)$ to (89 ± 27) keV.

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I. INTRODUCTION

In 2003, the BaBar Collaboration discovered a positive-parity scalar charm-strange meson $D_{s0}^*(2317)$ with a very narrow width [1]. The state was confirmed later by the CLEO Collaboration [2]. The discovery of this state has inspired heated discussions in the past decade. The key point is to understand the low mass of this state, which is more than 100 MeV lower than the prediction for the lowest scalar $c\bar{s}$ meson in, for instance, the Godfrey-Isgur quark model [3]. There are several interpretations of its structure, such as being a DK molecule, the chiral partner of the pseudoscalar D_s , a conventional $c\bar{s}$ state, coupled-channel effects between the $c\bar{s}$ state and DK continuum etc. For a detailed review of the properties and the phenomenology of these states see Ref. [4]. In order to distinguish them, one has to explore the consequences of each interpretation, and identify quantities which have different values in different interpretations. Arguably the most promising quantity is the isospin breaking width $\Gamma(D_{s0}^*(2317) \rightarrow D_s\pi)$. It is of order 10 keV if the $D_{s0}^*(2317)$ is a $c\bar{s}$ meson [5, 6], while it is of order 100 keV [7–9] in the DK molecular picture due to its large coupling to DK and the proximity of the DK threshold. Thus, the study of DK interaction is very important in order to understand the structure of $D_{s0}^*(2317)$. Although a direct simulation of the $DK(I=0)$ channel suffers from disconnected diagrams, one may obtain useful information on the DK interaction by calculating the scattering lengths of the disconnected-diagram-free channels which can be related to $DK(I=0)$ through SU(3) flavor symmetry. This is the strategy we will follow in this paper.

Lattice QCD calculations of the properties of hadronic interactions such as elastic scattering phases shifts and scattering lengths have recently started to develop. Precision results have been obtained in the light meson sector for certain processes such as pion-pion, kaon-kaon and pion-kaon scattering and preliminary results for baryon-baryon scattering lengths have been presented. A review of these calculations can be found in Ref. [10]. In the heavy meson sector, only a few calculations have been done, including quenched calculations in Refs. [11, 12] and calculations in full QCD in Refs. [13, 14]. In this work, we study scattering processes where one of the hadrons contains a charm quark in full lattice QCD.

Extracting hadronic interactions from Lattice QCD calculations is not straightforward due to the Maiani-Testa theorem [15], which states that the S -matrix can not be extracted from infinite-volume Euclidean-space Green functions except at kinematic thresholds. However, this problem can be evaded by computing the correlation functions in a finite volume. Lüscher has shown that one can obtain the scattering amplitude from the energy of two particles in a finite volume [16, 17].

We use Lüscher’s finite volume technique to calculate the scattering lengths. We then use unitarized chiral perturbation theory to extrapolate our results to the physical pion mass. Having fitted the appearing low-energy constants (LECs) to the lattice data, we are also able to make predictions for other channels, in particular for the isospin zero, strangeness one channel in which the $D_{s0}^*(2317)$ resides.

The paper is organized as follows. The lattice formulation of the light and heavy quark actions will be discussed in Section II. Lüscher’s formula will be briefly introduced in Section III. The numerical results for the scattering lengths of five channels $D\bar{K}(I=0)$, $D\bar{K}(I=1)$, $D_s K$, $D\pi(I=3/2)$ and $D_s\pi$ which are free of disconnected diagrams will be given in Section IV. Chiral extrapolations will be performed using unitarized chiral perturbation theory in Section V, and values of the LECs in the chiral Lagrangian will be determined. Predictions for other channels using the LECs are given Section VI, and in particular, implications on the $D_{s0}^*(2317)$ will be discussed. The last section is devoted to a brief summary.

II. LATTICE FORMULATION

A. Light-Quark Action

In this work we employ the “coarse” ($a \simeq 0.125$ fm) gauge configurations generated by the MILC Collaboration [18] using the one-loop tadpole-improved gauge action [19], where both $\mathcal{O}(a^2)$ and $\mathcal{O}(g^2 a^2)$ errors are removed. For the fermions in the vacuum, the asqtad-improved Kogut-Susskind (staggered) action [20–25] is used. This is the so-called Naik action [26] ($\mathcal{O}(a^2)$ improved Kogut-Susskind action) with smeared links for the one-link terms so that couplings to gluons with any of their momentum components equal to π/a are set to zero, resulting in a reduction of the flavor symmetry violations present in the Kogut-Susskind action.

For the valence light quarks (up, down and strange) we use the five-dimensional Shamir [27, 28] domain-wall fermion action [29]. The domain-wall fermion action introduces a fifth dimension of extent L_5 and a mass parameter M_5 ; in our case, the values $L_5 = 16$ and $M_5 = 1.7$, both in lattice units, were chosen. The physical quark fields, $q(\vec{x}, t)$, reside on the 4-dimensional boundaries of the fifth coordinate. The left and right chiral components are separated on the corresponding boundaries, resulting in an action with chiral symmetry at finite lattice spacing as $L_5 \rightarrow \infty$. We use hypercubic-smeared gauge links [30–33] to minimize the residual chiral symmetry breaking, and the bare quark-mass parameter $(am)_q^{\text{dwf}}$ is introduced as a direct coupling of the boundary

Ensemble	β	am_l	am_s	am_l^{dwf}	am_s^{dwf}	N_{cfgs}	N_{props}
m007	6.76	0.007	0.050	0.0081	0.081	461	2766
m010	6.76	0.010	0.050	0.0138	0.081	636	3816
m020	6.79	0.020	0.050	0.0313	0.081	480	1920
m030	6.81	0.030	0.050	0.0478	0.081	563	1689

TABLE I: The parameters of the configurations and domain-wall propagators used in this work. The subscript l denotes light quark, and s denotes the strange quark. The superscript “dwf” denotes domain-wall fermion.

chiral components. The light quark propagators were provided to us by the NPLQCD [10] and LHP [34–36] Collaborations.

The calculation we have performed, because the valence and sea quark actions are different, is inherently partially quenched and therefore violates unitarity. Unlike conventional partially quenched calculations, to restore unitarity, one must take the continuum limit in addition to tuning the valence and sea quark masses to be degenerate. This process is aided by the use of mixed-action chiral perturbation theory [37–42]. Given the situation, there is an ambiguity in the choice of the valence light-quark masses. One appealing choice is to tune the valence light quark masses such that the valence pion mass is degenerate with the Goldstone staggered pion mass. In the continuum limit, the $N_f = 2$ staggered action has an $SU(8)_L \otimes SU(8)_R \otimes U(1)_V$ chiral symmetry due to the four-fold taste degeneracy of each flavor, and each pion has 15 degenerate partners. At finite lattice spacing this symmetry is broken and the taste multiplets are no longer degenerate, but have splittings that are $\mathcal{O}(\alpha_s^2 a^2)$ [20–22, 25, 43]. The propagators used in this work were tuned to give valence pions that match the Goldstone Kogut-Susskind pion. This is the only pion that becomes massless in the chiral limit at finite lattice spacing. As a result of this choice, the valence pions are as light as possible, while being tuned to one of the staggered pion masses, providing better convergence in the chiral perturbation theory needed to extrapolate the lattice results to the physical quark-mass point. This set of parameters, listed in Table I, was first used by LHPC [34, 35] and recently utilized to compute the spectroscopy of hadrons composed of up, down and strange quarks [36].

B. Heavy-Quark Action

For the charm quark we use a relativistic heavy quark action motivated by the Fermilab approach [44]. This action controls discretization errors of $\mathcal{O}((am_Q)^n)$. Following the Symanzik improvement [45], an effective continuum action is constructed using operators that are invariant under discrete rotations, parity-reversal and charge-conjugation transformations, representing the long-distance limit of our lattice theory, including leading finite- a errors. Using only the Dirac operator and the gluon field tensor (and distinguishing between the time and space components of each), we enumerate seven operators with dimension up to five. By applying the isospectral transformations [46], the redundant operators are identified and their coefficients are set to appropriate convenient values. The lattice action then takes the form

$$S = S_0 + S_B + S_E, \quad (1)$$

with

$$S_0 = \sum_x \bar{Q}(x) \left[m_0 + \left(\gamma_0 \nabla_0 - \frac{a}{2} \Delta_0 \right) + \nu \sum_i \left(\gamma_i \nabla_i - \frac{a}{2} \Delta_i \right) \right] Q(x), \quad (2)$$

$$S_B = -\frac{a}{2} c_B \sum_x \bar{Q}(x) \left(\sum_{i < j} \sigma_{ij} F_{ij} \right) Q(x), \quad (3)$$

$$S_E = -\frac{a}{2} c_E \sum_x \bar{Q}(x) \left(\sum_i \sigma_{0i} F_{0i} \right) Q(x), \quad (4)$$

where the operator $Q(x)$ annihilates a heavy quark field, a is the lattice spacing, ∇_0 and ∇_i are first-order lattice derivatives in the time and space directions, Δ_0 and Δ_i are second-order lattice derivatives, and $F_{\mu\nu}$ is the gauge field strength tensor. The spectrum of heavy-quark bound states can be determined accurately through $|\vec{p}|a$ and $(am_Q)^n$ for arbitrary exponent n by using a lattice action containing m_0 , ν , c_B and c_E , which are functions of am_Q .

The coefficients c_B and c_E are different due to the broken space-time interchange symmetry, which can be computed in perturbation theory by requiring elimination of the heavy-quark discretization errors at a given order in the strong coupling constant α_s . We use the tree-level tadpole-improved results obtained by using field transformation (as in Ref. [46]):

$$c_B = \frac{\nu}{u_0^3}, \quad c_E = \frac{1}{2}(1 + \nu) \frac{1}{u_0^3}, \quad (5)$$

where u_0 is the tadpole factor

$$u_0 = \left\langle \frac{1}{3} \sum_p \text{Tr}(U_p) \right\rangle^{1/4}, \quad (6)$$

and U_p is the product of gauge links around the fundamental lattice plaquette p . The remaining two parameters m_0 and ν are determined nonperturbatively. The bare charm-quark mass m_0 is tuned so that the experimentally observed spin average of the J/ψ and η_c masses

$$M_{\text{avg}} = \frac{1}{4}M_{\eta_c} + \frac{3}{4}M_{J/\psi} \quad (7)$$

is reproduced. For each ensemble, we calculate M_{avg} at two charm-quark masses (denoted $m_1 = 0.2034$ and $m_2 = 0.2100$) and linearly extrapolate it to the experimental value to determine the parameter $m_0 = m_c^{\text{phys}}$. The value of ν must be tuned to restore the dispersion relation $E_h^2 = m_h^2 + c^2 p^2$ such that $c^2 = 1$. To do this, we calculate the single-particle energy of η_c , J/ψ , D_s and D at the six lowest momenta (with unit of a^{-1}): $(2\pi/L)(0, 0, 0)$, $(2\pi/L)(1, 0, 0)$, $(2\pi/L)(1, 1, 0)$, $(2\pi/L)(1, 1, 1)$, $(2\pi/L)(2, 0, 0)$, $(2\pi/L)(2, 1, 0)$. For each ensemble, the energy levels are calculated at the two charm-quark masses (m_1 and m_2) and extrapolated to the physical charm-quark mass m_c^{phys} . The values of c^2 are obtained by fitting the extrapolated energy levels to the dispersion relation. We tune ν using the dispersion relation of η_c . The dispersion relations for either the charmonium J/ψ or the charm-light mesons (D and D_s) are generally consistent with $c^2 = 1$ within 1-2%. Since the values of ν and m_0 are coupled, one needs to iterate the tuning process in order to achieve a consistent pair of values. For the details of tuning the bare charm-quark mass m_0 and the value of ν , see reference [47].

III. LÜSCHER'S FORMULA

Lüscher has shown that the scattering phase shift is related to the energy shift (ΔE) in the total energy of two interacting hadrons in a finite box [16, 17].

The center-of-mass momentum p can be obtained by the relation

$$\Delta E = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} - m_1 - m_2, \quad (8)$$

where m_1 and m_2 are the rest masses of the two hadrons.

To obtain $p \cot \delta(p)$, where $\delta(p)$ is the phase shift, we use the formula [48]

$$p \cot \delta(p) = \frac{1}{\pi L} \mathbf{S} \left(\left(\frac{pL}{2\pi} \right)^2 \right), \quad (9)$$

where the \mathbf{S} function is defined as

$$\mathbf{S}(x) = \sum_{\mathbf{j}}^{\|\mathbf{j}\| < \Lambda} \frac{1}{\|\mathbf{j}\|^2 - x} - 4\pi\Lambda. \quad (10)$$

The sum is over all three-vectors of integers \mathbf{j} such that $|\mathbf{j}| < \Lambda$, and the limit $\Lambda \rightarrow \infty$ is implicit.

If the interaction range is much smaller than the lattice size, $p \cot \delta(p)$ is given by

$$p \cot \delta(p) = \frac{1}{a} + \mathcal{O}(p^2), \quad (11)$$

where a is the scattering length (not to be confused with the lattice spacing which has the same notation and dimension). Note that we take the sign convention that a repulsive interaction has a negative scattering length. The higher order terms in Eq. (11) can be ignored if the effective range of the interaction is much smaller than the length scale associated to the center-of-mass momentum p . If we ignore the higher order terms, the scattering length can be calculated by

$$a = \pi L \mathbf{S}^{-1} \left(\left(\frac{pL}{2\pi} \right)^2 \right). \quad (12)$$

IV. NUMERICAL RESULTS

In the following, we list all the channels we study. The interpolating operators for these two particle states are

$$\begin{aligned} \mathcal{O}_{D_s \pi} &= D_s^- \pi^+, & \mathcal{O}_{D\pi}^{I=3/2} &= D^+ \pi^+, & \mathcal{O}_{D_s K} &= D_s^+ K^+, \\ \mathcal{O}_{D\bar{K}}^{I=1} &= D^+ \bar{K}^0, & \mathcal{O}_{D\bar{K}}^{I=0} &= D^+ K^- - D^0 \bar{K}^0, \end{aligned}$$

where D_s^- , D_s^+ , D^+ , K^0 , K^- , K^+ and π^+ are the operators for one particle states, the subscripts π , D , K and \bar{K} represent the isospin triplet (π^+ , π^0 , π^-) and doublets (D^+ , D^0), (K^+ , K^0) and (\bar{K}^0 , K^-), respectively.

The total energy of two interacting hadrons (h_1 and h_2) is obtained from the four-point correlation function:

$$G^{h_1 h_2}(t) = \langle \mathcal{O}_{h_1 h_2}(t)^\dagger \mathcal{O}_{h_1 h_2}(0) \rangle. \quad (13)$$

To extract the energy shift ΔE , we define a ratio $R^{h_1 h_2}(t)$:

$$R^{h_1 h_2}(t) = \frac{G^{h_1 h_2}(t)}{G^{h_1}(t) G^{h_2}(t)} \longrightarrow \exp(-\Delta E \cdot t), \quad (14)$$

where $G^{h_1}(t, 0)$ and $G^{h_2}(t, 0)$ are two-point functions. ΔE is obtained by fitting $R^{h_1 h_2}(t)$ to a single exponential in the region where the effective mass exhibits a plateau.

For each channel, we calculate the ratio $R^{h_1 h_2}$ at two different charm quark masses and four different light valence quark masses. Figure 1 shows the effective energy shifts of each channel calculated from ensemble m007 at the bare charm-quark mass $m_2 = 0.2100$. The fitted energy

shifts and the fitting ranges are indicated by the grey bars in these plots. The heights of the grey bars show the statistical errors. The effective energy shift plots for other ensembles are similar.

The energy shifts are linearly extrapolated to the physical charm-quark mass on each ensemble.

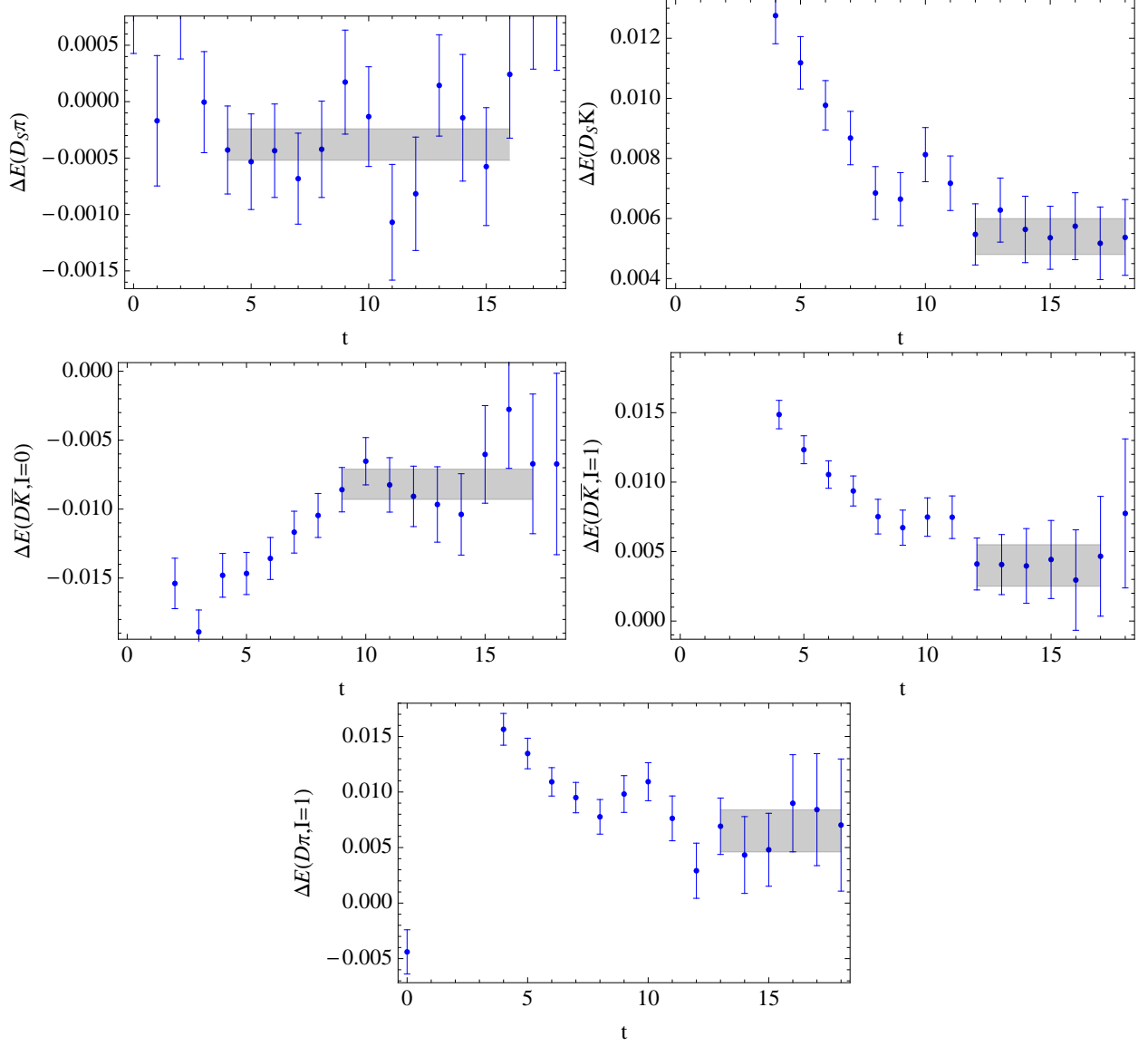


FIG. 1: Effective energy shifts plots of the scattering channels $D_s\pi$, D_sK , $D\bar{K}(I = 0)$, $D\bar{K}(I = 1)$, $D\pi(I = 3/2)$. All plots are for ensemble m007. The grey bars show the fitted energy shifts and the fitting ranges. The height of the grey bars show the statistical errors.

V. CHIRAL EXTRAPOLATIONS OF THE SCATTERING LENGTHS

Because the simulations are performed at unphysical quark masses, chiral extrapolation is necessary in order to obtain the values of scattering lengths at the physical quark masses. There have been calculations based on chiral Lagrangians for these scattering lengths [49–52]. They were first calculated in Ref. [49] using a unitarized chiral approach. The basic observation is because of the coupled-channel effect and the large kaon mass, the interaction of some of the channels is so strong that a nonperturbative treatment is necessary, and in one channel even a bound state is produced. The method was followed up recently in Ref. [52]. Some other authors treated the interaction perturbatively, and calculated the scattering lengths up to leading one-loop order in chiral perturbation theory with [50] and without [51] a heavy quark expansion. Here we take the same route as Ref. [49], and resum the chiral amplitude up to next-to-leading order, which is $\mathcal{O}(p^2)$. The resummed amplitude in the on-shell approximation reads [53–55]

$$T(s) = V(s)[1 - G(s)V(s)]^{-1}, \quad (15)$$

where $V(s)$ is the S -wave projection of the $\mathcal{O}(p^2)$ scattering amplitude, and $G(s)$ is the scalar loop function regularized by a subtraction constant $\tilde{a}(\lambda)$

$$G(s) = \frac{1}{16\pi^2} \left\{ \tilde{a}(\lambda) + \ln \frac{m_2^2}{\lambda^2} + \frac{m_1^2 - m_2^2 + s}{2s} \ln \frac{m_1^2}{m_2^2} + \frac{\sigma}{2s} [\ln(s - m_1^2 + m_2^2 + \sigma) - \ln(-s + m_1^2 - m_2^2 + \sigma) + \ln(s + m_1^2 - m_2^2 + \sigma) - \ln(-s - m_1^2 + m_2^2 + \sigma)] \right\}, \quad (16)$$

with $\sigma = \{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\}^{1/2}$. λ is the scale of dimensional regularization, and a change of λ can be absorbed by a corresponding change of $\tilde{a}(\lambda)$. The value $\lambda = 1$ GeV will be taken in the following. Promoting $T(s)$, $V(s)$ and $G(s)$ to be matrix-valued quantities, it is easy to generalize Eq. (15) to coupled channels.

Using the $\mathcal{O}(p^2)$ chiral Lagrangian constructed in Ref. [9], the scattering amplitudes are given by

$$V(s, t, u) = \frac{1}{F^2} \left[\frac{C_{\text{LO}}}{4}(s - u) - 4C_0h_0 + 2C_1h_1 - 2C_{24}H_{24}(s, t, u) + 2C_{35}H_{35}(s, t, u) \right], \quad (17)$$

where F is the pion decay constant in the chiral limit, and the coefficients C_i can be found in Table II. Further,

$$H_{24}(s, t, u) = 2h_2p_2 \cdot p_4 + h_4(p_1 \cdot p_2p_3 \cdot p_4 + p_1 \cdot p_4p_2 \cdot p_3),$$

and

$$H_{35}(s, t, u) = h_3p_2 \cdot p_4 + h_5(p_1 \cdot p_2p_3 \cdot p_4 + p_1 \cdot p_4p_2 \cdot p_3).$$

(S, I)	Channels	C_{LO}	C_0	C_1	C_{24}	C_{35}
$(-1, 0)$	$D\bar{K} \rightarrow D\bar{K}$	-1	M_K^2	M_K^2	1	-1
$(-1, 1)$	$D\bar{K} \rightarrow D\bar{K}$	1	M_K^2	$-M_K^2$	1	1
$(2, \frac{1}{2})$	$D_s K \rightarrow D_s K$	1	M_K^2	$-M_K^2$	1	1
$(0, \frac{3}{2})$	$D\pi \rightarrow D\pi$	1	M_π^2	$-M_\pi^2$	1	1
$(1, 1)$	$D_s \pi \rightarrow D_s \pi$	0	M_π^2	0	1	0
	$DK \rightarrow DK$	0	M_K^2	0	1	0
	$DK \rightarrow D_s \pi$	1	0	$-(M_K^2 + M_\pi^2)/2$	0	1
$(1, 0)$	$DK \rightarrow DK$	-2	M_K^2	$-2M_K^2$	1	2
	$D_s \eta \rightarrow D_s \eta$	0	M_η^2	$-2M_\eta^2 + 2M_\pi^2/3$	1	$\frac{4}{3}$
	$DK \rightarrow D_s \eta$	$-\sqrt{3}$	0	$-\sqrt{3}(5M_K^2 - 3M_\pi^2)/6$	0	$\frac{1}{\sqrt{3}}$
$(0, \frac{1}{2})$	$D\pi \rightarrow D\pi$	-2	M_π^2	$-M_\pi^2$	1	1
	$D\eta \rightarrow D\eta$	0	M_η^2	$-M_\pi^2/3$	1	$\frac{1}{3}$
	$D_s \bar{K} \rightarrow D_s \bar{K}$	-1	M_K^2	$-M_K^2$	1	1
	$D\eta \rightarrow D\pi$	0	0	$-M_\pi^2$	0	1
	$D_s \bar{K} \rightarrow D\pi$	$-\frac{\sqrt{6}}{2}$	0	$-\sqrt{6}(M_K^2 + M_\pi^2)/4$	0	$\frac{\sqrt{6}}{2}$
	$D_s \bar{K} \rightarrow D\eta$	$-\frac{\sqrt{6}}{2}$	0	$-\sqrt{6}(5M_K^2 - 3M_\pi^2)/12$	0	$-\frac{1}{\sqrt{6}}$

TABLE II: The coefficients in the scattering amplitudes $V(s, t, u)$. The channels are labelled by strangeness (S) and isospin (I).

Note that the term $h_1 \tilde{\chi}_+ = h_1(\chi_+ - \langle \chi_+ \rangle/3)$ in the Lagrangian in Refs. [9, 49] has been replaced by $h_1 \chi_+$, which amounts to a redefinition of h_0 (for the details of the Lagrangian and the definition of χ_+ , we refer to Refs. [9, 49]). In this way, the h_1 term does not contain the $1/N_c$, with N_c being the number of colors, suppressed part $\langle \chi_+ \rangle$ any more. This was also done in Ref. [52].

In previous works [9, 49], the large- N_c suppressed low-energy constants (LECs) $h_{0,2,4}$ were dropped to reduce the number of parameters. However, when fitting to the lattice data at several unphysical quark masses, this is no longer necessary. In this work, we will keep all of the LECs at this order, and fit them to the lattice data. This strategy were also taken in Refs. [50–52], where the preliminary lattice results [13] were used. By definition, the LECs are independent of the pion mass. We further need to assume that the subtraction constant is the same for various channels, and neglect its pion mass dependence. In principle, this assumption is not necessary for a unitarization procedure matched to the full one-loop level of the perturbative calculation [55, 56], which will be left for the future.

From the SU(3) mass splitting of the charmed mesons, the value of h_1 is fixed to be $h_1 = 0.42$.

	$D\bar{K}(I=1)$	$D\bar{K}(I=0)$	$D_s K$	$D\pi(I=3/2)$	$D_s \pi$	M_π	a (fm)
m007	-1.19(0.40)	5.34(1.45)	-1.58(0.14)	-1.16(0.30)	0.08(0.04)	0.1842	0.1207
m010	-1.89(0.12)	6.21(1.04)	-1.55(0.09)	-1.38(0.10)	0.08(0.03)	0.2238	0.1214
m020	-1.49(0.25)	4.43(1.33)	-1.40(0.20)	-1.08(0.30)	0.13(0.05)	0.3113	0.1202
m030	-1.59(0.13)	7.46(1.56)	-1.67(0.10)	-1.68(0.13)	0.32(0.05)	0.3752	0.1200

TABLE III: The values of scattering lengths for five channels and M_π in lattice units. The values of the lattice spacing a are also given in the last column.

We still have six parameters, which are $\tilde{a}, h_3, h_5, h_0, h_2$ and h_4 . They are to be fitted to the lattice data. However, there is a high correlation between h_3 and h_5 , as well as a similar correlation between h_2 and h_4 . In the heavy quark limit, the S -wave projected amplitudes cannot distinguish the $h_{4(5)}$ terms from the $h_{2(3)}$ ones [57]. Hence, we may reduce the correlations largely by rewriting $H_{24}(s, t, u)$ and $H_{35}(s, t, u)$ as

$$H_{24}(s, t, u) = 2h_{24}p_2 \cdot p_4 + h_4 (p_1 \cdot p_2 p_3 \cdot p_4 + p_1 \cdot p_4 p_2 \cdot p_3 - 2\bar{M}_D^2 p_2 \cdot p_4),$$

and

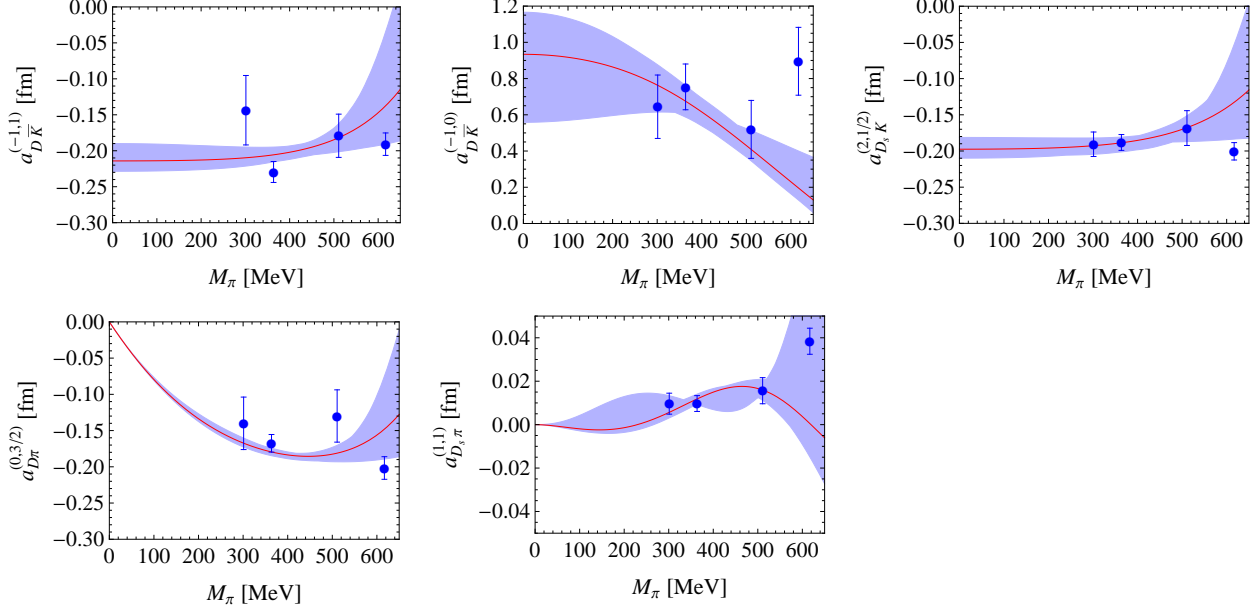
$$H_{35}(s, t, u) = h_{35}p_2 \cdot p_4 + h_5 (p_1 \cdot p_2 p_3 \cdot p_4 + p_1 \cdot p_4 p_2 \cdot p_3 - 2\bar{M}_D^2 p_2 \cdot p_4),$$

where $\bar{M}_D \equiv (M_D^{\text{phy}} + M_{D_s}^{\text{phy}})/2$, the average of the masses of the D and D_s at the physical pion mass, is introduced to match the dimensions. The new parameters h_{24} and h_{35} are dimensionless, and their relations to the old ones are $h_{24} = h_2 + h'_4$ and $h_{35} = h_3 + 2h'_5$, where $h'_4 = h_4 \bar{M}_D^2$ and $h'_5 = h_5 \bar{M}_D^2$.

There are four different light quark masses in our data set, corresponding to the four ensembles (m007, m010, m020 and m030) with pion masses approximately 301 MeV, 364 MeV, 511 MeV and 617 MeV, respectively. There are in total 20 data points in the five channels. The values of the scattering lengths for all the channels are collected in Table III, where the values of M_π and lattice spacings for all the ensembles are also given. However, for a pion mass as high as 617 MeV, the kaon mass would be even higher, around 700 MeV. Such values are too large for a controlled chiral expansion. Therefore, we will only fit to the ensembles m007, m010 and m020. To minimize the contamination from a particular scale-setting method, we fit to the dimensionless product of the pion mass and the scattering length. The fit was performed using the FORTRAN package MINUIT [58]. The best fit has $\chi^2/\text{dof} = 1.17$, and the resulting parameters are collected in Table IV, where the asymmetric 1σ uncertainties are calculated using the MINOS algorithm in

Fitting range	χ^2/dof	$\tilde{a}(\lambda = 1 \text{ GeV})$	h_0	h_{24}	h'_4	h_{35}	h'_5
m007-m020	1.17	$-1.53^{+0.15}_{-0.38}$	$0.49^{+3.51}_{-0.44}$	$-0.44^{+0.36}_{-3.65}$	$-0.23^{+0.49}_{-0.68}$	$0.18^{+0.15}_{-0.21}$	$-1.62^{+1.40}_{-1.09}$

TABLE IV: Results of fitting to the lattice data of the scattering lengths with 6 parameters.

FIG. 2: Fit to the data of the scattering lengths corresponding to ensembles m007-m020 in each channel. The superscript (S, I) is the (strangeness, isospin) for each channel.

MINUIT. A comparison of the best fit and the lattice data is shown in Fig. 2, where the solid curves correspond to the results of the best fit, and the bands reflect the uncertainties propagated from the lattice data. At the physical pion mass, the extrapolated scattering lengths for the five channels are presented in Table V.

One can see that all the dimensionless parameters have a natural size, i.e., the absolute values of $h_{0,24,35}$ and $h'_{3,5}$ are of order unity. However, with the current fit, some of the parameters are not well constrained. For instance, one of the asymmetric uncertainties of either h_0 or h_{24} is much larger than its central value. It turns out that when h_0 takes a larger value, h_{24} will take a smaller value, and \tilde{a} will also take a smaller value. The asymmetry in the uncertainties of these three parameters is larger than that for the others. This indicates a strong correlation of these parameters. Furthermore, for a large part of the uncertainty range, h_0 is larger than h_1 . This is in contrast to the N_c counting which suggests an opposite ordering. As will be shown in Section VIB, the parameters can be much better constrained with an additional assumption.

When performing the fit, we have used the physical value for the pion decay constant $F =$

Channels	$D\bar{K}(I=1)$	$D\bar{K}(I=0)$	$D_s K$	$D\pi(I=3/2)$	$D_s \pi$
a (fm)	$-0.21^{+0.02}_{-0.01}$	$0.90^{+0.20}_{-0.32}$	$-0.20^{+0.02}_{-0.01}$	$-0.103^{+0.005}_{-0.002}$	$-0.002^{+0.010}_{-0.002}$

TABLE V: The scattering lengths extrapolated to the physical light quark masses.

Channels	$D\pi(I=1/2)$	$DK(I=1)$	$D_s \bar{K}$
a (fm)	$0.31^{+0.08}_{-0.02}$	$0.09^{+0.08}_{-0.06} + i(0.15^{+0.05}_{-0.12})$	$-0.06^{+0.05}_{-0.90} + i(0.47^{+0.22}_{-0.34})$

TABLE VI: Scattering lengths of $D\pi(I=1/2)$, $DK(I=1)$ and $D_s K$ at the physical pion mass predicted from the fit.

92.21 MeV [59]. The difference from the chiral limit value and hence its pion mass dependence is a higher order effect, and is neglected here, although it might have some influence. Furthermore, we have expressed the masses of the other hadrons in terms of masses in the SU(2) chiral limit, and the pion mass. The kaon mass is written as $M_K = \mathring{M}_K + M_\pi^2/(4\mathring{M}_K)$ (see, for instance, [60]), and the charmed meson masses are

$$M_D = \mathring{M}_D + (h_1 + 2h_0) \frac{M_\pi^2}{\mathring{M}_D}, \quad M_{D_s} = \mathring{M}_{D_s} + 2h_0 \frac{M_\pi^2}{\mathring{M}_{D_s}}. \quad (18)$$

VI. IMPLICATIONS FOR OTHER CHANNELS

A. Scattering lengths

In this work we did not calculate the scattering lengths on the lattice of the channels whose Wick contractions involve disconnected diagrams due to the computational difficulties, as well as the additional lattice artifacts present in these channels due to the use of Kogut-Susskind sea quarks. However, once we have determined the LECs in the chiral Lagrangian, we can make predictions on the scattering lengths of these channels. The results for the scattering lengths of $D\pi(I=1/2)$, $DK(I=1)$ and $D_s K$ at the physical pion mass are presented in Table VI. For $DK(I=1)$, the imaginary part of the scattering length originates because it couples to $D_s \pi$ with a lower threshold. Similarly, $D_s K$ couples to $D\pi$ and $D\eta$ so that the scattering length is complex, too. The result for the $D\pi(I=1/2)$ channel is consistent with the indirect extraction from lattice calculations of the $D\pi$ scalar form factor (0.41 ± 0.06) fm [61]. At a pion mass of about 266 MeV, our prediction is $(0.90^{+0.83}_{-0.19})$ fm, compatible with the very recent full QCD calculation (0.81 ± 0.14) fm [14].

Fitting range	χ^2/dof	h_0	h_{24}	h'_4	h_{35}	h'_5
m007-m020	1.20	$-0.09^{+0.08}_{-0.06}$	$-0.06^{+0.03}_{-0.04}$	$-1.59^{+0.60}_{-0.77}$	$-0.15^{+0.09}_{-0.13}$	$-3.11^{+1.16}_{-1.83}$

TABLE VII: Results of fitting to the lattice data of the scattering lengths with 5 parameters. The subtraction constant is solved from fixing the pole in the $(S, I) = (1, 0)$ channel to 2317.8 MeV.

The most interesting channel is the one with $(S, I) = (1, 0)$, where the $D_{s0}^*(2317)$ resides. This state was proposed to be a hadronic molecule with a dominant DK component by several groups [62–66]. The attraction in this channel is so strong that a pole emerges in the resummed amplitude. However, as can be seen from Table IV, the uncertainties of some of the parameters are rather large, which causes a large range of the pole position. The pole can be a bound state pole in the first Riemann sheet at around 2200 MeV, and it can also be a resonance pole in the second Riemann sheet. As a result, the scattering length in this channel ranges from $-\infty$ to $+\infty$. A large value of positive scattering length implies the existence of a virtual state, which is a pole on the real axis on the second Riemann sheet, while a large value of negative scattering length (the absolute value is large) implies a bound state close to threshold. As emphasized in, for instance, Refs. [67, 68], if there is an S -wave shallow bound state, the scattering length is related to the binding energy, and to the wave function renormalization constant Z , with $(1 - Z)$ being the probability of finding the molecular component in the physical state (for $Z = 0$, the physical state is purely a bound state). The relation reads

$$a = -2 \left(\frac{1 - Z}{2 - Z} \right) \frac{1}{\sqrt{2\mu\epsilon}} \left(1 + \mathcal{O}(\sqrt{2\mu\epsilon}/\beta) \right), \quad (19)$$

where μ and ϵ are the reduced mass and binding energy, respectively. Corrections of the above equation come from neglecting the range of forces, $1/\beta$, which contains information of the $D_s\eta$ channel. Were the $D_{s0}^*(2317)$ a pure DK bound state, the value of $DK(I = 0)$ scattering length would be $a = -1.05$ fm.

B. Isospin breaking width of the $D_{s0}^*(2317)$

In the following, we will assume that the $D_{s0}^*(2317)$ corresponds to the pole generated in the $(S, I) = (1, 0)$ channel, and explore the implications of our lattice calculation on this state. We will fix the pole position to the mass of the $D_{s0}^*(2317)$, 2317.8 MeV [59], on the first Riemann sheet. We fit the lattice results of the scattering lengths with h_0 , h_{24} , h_{35} , h'_4 and h'_5 , and adjust the subtraction constant $\tilde{a}(\lambda = 1 \text{ GeV})$ to reproduce the mass of the $D_{s0}^*(2317)$. Again, we only

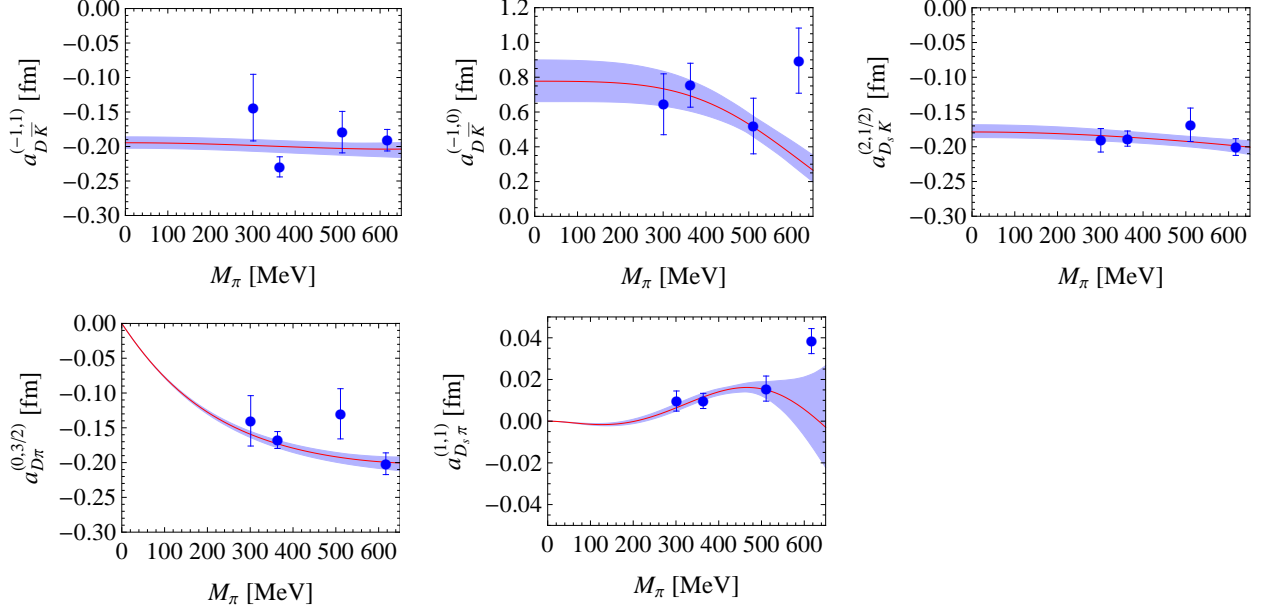


FIG. 3: Fit to the data corresponding to ensembles m007-m020 in each channel with 5 parameters. The subtraction constant is solved from fixing the pole in the $(S, I) = (1, 0)$ channel to 2317.8 MeV.

Channels	$D\bar{K}(I=1)$	$D\bar{K}(I=0)$	$D_s K$	$D\pi(I=3/2)$	$D_s \pi$
$a(\text{fm})$	$-0.20(1)$	$0.77(12)$	$-0.18(1)$	$-0.099(2)$	$-0.002(1)$

TABLE VIII: The scattering lengths extrapolated to the physical light quark masses from the 5-parameter fit.

fit to the ensembles m007, m010 and m020. The best fit gives $\chi^2/\text{dof} = 1.20$, which is almost the same as the one with one more parameter in Section V. The parameter values together with the 1σ statistical uncertainties are given in Table VII. The fitted results are presented in Fig. 3. At the physical pion mass, the extrapolated values of the scattering lengths are listed in Table VIII.

With the newly fitted parameters, the scattering lengths for several other channels are predicted, and the results are listed in Table IX. One sees that the value for the $DK(I=0)$ channel is close to the result of Eq. (19), -1.05 fm, with $Z=0$. The derivation from this value is partly due to the

Channels	$D\pi(I=1/2)$	$DK(I=0)$	$DK(I=1)$	$D_s \bar{K}$
$a(\text{fm})$	0.37 ± 0.01	$-0.85^{+0.07}_{-0.05}$	$0.03^{+0.03}_{-0.05} + i(0.05 \pm 0.04)$	$-0.19 \pm 0.05 + i(0.12^{+0.09}_{-0.06})$

TABLE IX: Scattering lengths of $D\pi(I=1/2)$, $DK(I=0)$, $DK(I=1)$ and $D_s \bar{K}$ at the physical pion mass predicted from the 5-parameter fit.

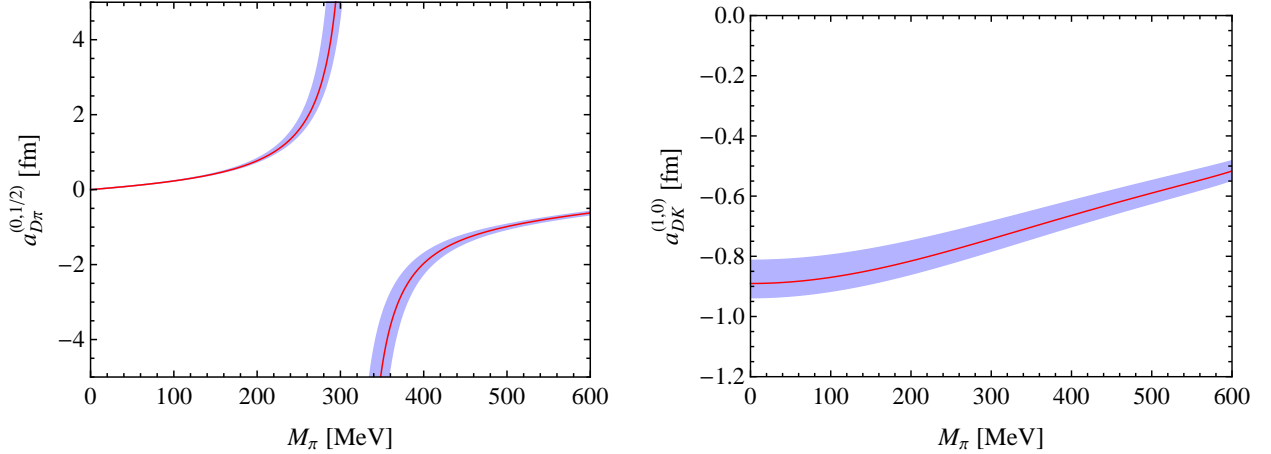


FIG. 4: Predicted pion mass dependence of the $D\pi(I = 1/2)$ and $DK(I = 0)$ scattering lengths using parameters from the 5-parameter fit. The solid curves are calculated using the parameters from the best fit, and the bands reflect the uncertainties.

coupled-channel $D_s\eta$, and partly due to the energy dependence in the interaction. Using Eq. (19), the value of Z is in the range $[0.25, 0.41]$. Therefore, the main component of the $D_{s0}^*(2317)$ is the isoscalar DK molecule.

The result for the $D\pi(I = 1/2)$ channel at the physical pion mass, with much smaller uncertainty, is still consistent with the indirect extraction, (0.41 ± 0.06) fm, in Ref. [61]. However, our prediction at a pion mass of about 266 MeV, $(2.17^{+0.56}_{-0.25})$ fm, is larger than (0.81 ± 0.14) fm obtained in [14]. From Fig. 4, one sees that such a pion mass is close to the transition point where the scattering length changes sign due to the generation of a pole (for more discussions, see [49]). In such a region, the value of the scattering length changes quickly. For instance, decreasing the pion mass to 220 MeV, one would get a much smaller value $(1.00^{+0.09}_{-0.05})$ fm.

From Table VII, one sees that all the dimensionless LECs are of natural size. The value of h_0 is much smaller than $h_1 = 0.42$. This is consistent with the N_c counting because the h_0 term is suppressed by $1/N_c$ as compared to the h_1 term. Furthermore, we also have the hierarchies $h'_4 < h'_5$ and $h_2 < h_3$ (recall that $h_2 = h_{24} - h'_4$ and $h_3 = h_{35} - 2h'_5$). For both cases, the left hand sides are suppressed by $1/N_c$ as compared to the right hand sides. This means the hadronic molecular assumption of the $D_{s0}^*(2317)$ is consistent with the naturalness and N_c counting.

All the above calculations have assumed the same mass for the up and down quarks, and neglected the electromagnetic interaction. This is the isospin symmetric case. However, the $D_{s0}^*(2317)$ decays into the isovector final state $D_s\pi$. In order to calculate this isospin breaking decay width, one has to take into account both the up and down quark mass difference and virtual photons. This

has been done in Ref. [9]. In Ref. [9], the N_c -suppressed operators, i.e. the h_0 , h_2 and h_4 terms have been dropped, and a somewhat arbitrarily chosen natural range $[-1, 1]$ was taken for h'_5 . The isospin breaking decay width was calculated to be $\Gamma(D_{s0}^*(2317) \rightarrow D_s\pi) = (180 \pm 110)$ keV [9]. With the values of all the h_i 's in Table VII, the result is updated to be

$$\Gamma(D_{s0}^*(2317) \rightarrow D_s\pi) = (89 \pm 27) \text{ keV}. \quad (20)$$

We have used the isospin breaking quark mass ratio $(m_d - m_u)/(m_s - \hat{m}) = 0.0299 \pm 0.018$, where $\hat{m} = (m_u + m_d)/2$, which is calculated using the lattice averages (up to end of 2011) of the light quark masses [69, 70].

VII. SUMMARY AND DISCUSSION

The low-energy interaction between a light pseudoscalar meson and a charmed pseudoscalar meson was studied. We have calculated scattering lengths of five channels $D\bar{K}(I=0)$, $D\bar{K}(I=1)$, D_sK , $D\pi(I=3/2)$ and $D_s\pi$ with four ensembles. Among these channels, the interaction of $D\bar{K}(I=0)$ is attractive, and that of the others is repulsive. The interaction of $D_s\pi$ is very weak, which is expected. The $D_s\pi$ and $DK(I=1)$ channels are mixed since they have the same quantum numbers. To perform a more reliable analysis of these two channels, we need to construct the correlation matrix and use the variational method to extract the energies of the two channels. The chiral extrapolation was performed using SU(3) unitarized chiral perturbation theory, and the LECs h_i 's in the chiral Lagrangian were determined from a fit to the lattice results. With the same set of parameters, we made predictions on other channels including $DK(I=0)$, $DK(I=1)$, $D\pi(I=1/2)$ and $D_s\bar{K}$. In particular, we found that the attractive interaction in the $DK(I=0)$ channel is strong enough so that a pole is generated in the unitarized scattering amplitude. The pole can be a bound state pole. By further fixing the pole in the $DK(I=0)$ channel to the $D_{s0}^*(2317)$, we found that the main component of the $D_{s0}^*(2317)$ is the isoscalar DK molecule. We revisited the isospin breaking decay width of the $D_{s0}^*(2317) \rightarrow D_s\pi$. The result (89 ± 27) keV updates the old result (180 ± 110) keV obtained in Ref. [9]. It is nice to see that the uncertainty of the width shrinks a lot.

It is possible to further constrain the values of h_i 's once simulations in other channels are done. Although a precise calculation of the other channels requires disconnected diagrams, one may obtain valuable information from the connected part only. The connected and disconnected parts can be calculated separately using partially quenched chiral perturbation theory (for reviews, see

Refs. [71, 72]), then a fit to the lattice calculation can be performed. This point has already been stressed, for instance, in Ref. [73] for the hadronic vacuum polarization and in Ref. [74] for the scalar form factor of the pion.

In our chiral extrapolation, the resummed chiral amplitude is of $\mathcal{O}(p^2)$. At this order, there is no counterterm to absorb the divergence of the loop function $G(s)$, because loops only start from $\mathcal{O}(p^3)$. As a result, we had to regularize the divergent loop by a subtraction constant, the pion-mass dependence of which was neglected. Were a full one-loop calculation available, the chiral amplitudes can be renormalized at one-loop order, and the representation of the pion mass dependence would be improved. However, more unknown LECs will be introduced in this way, and it is difficult to perform a fit with all of them to the present data. As mentioned above, more data can come from calculating the other channels, which is useful even if the disconnected contribution is neglected. Such a study with an improved chiral extrapolation is relegated to the future.

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- [1] B. Aubert et al. (BABAR Collaboration), Phys.Rev.Lett. **90**, 242001 (2003), hep-ex/0304021.
 - [2] D. Besson et al. (CLEO Collaboration), Phys.Rev. **D68**, 032002 (2003), hep-ex/0305100.
 - [3] S. Godfrey and N. Isgur, Phys.Rev. **D32**, 189 (1985).
 - [4] S.-L. Zhu, Int.J.Mod.Phys. **E17**, 283 (2008), hep-ph/0703225.

- [5] S. Godfrey, Phys.Lett. **B568**, 254 (2003), hep-ph/0305122.
- [6] P. Colangelo and F. De Fazio, Phys.Lett. **B570**, 180 (2003), hep-ph/0305140.
- [7] A. Faessler, T. Gutsche, V. E. Lyubovitskij, and Y.-L. Ma, Phys.Rev. **D76**, 014005 (2007), 0705.0254.
- [8] M. F. Lutz and M. Soyeur, Nucl.Phys. **A813**, 14 (2008), 0710.1545.
- [9] F.-K. Guo, C. Hanhart, S. Krewald, and U.-G. Meißner, Phys.Lett. **B666**, 251 (2008), 0806.3374.
- [10] S. R. Beane, K. Orginos, and M. J. Savage, Int.J.Mod.Phys. **E17**, 1157 (2008), 0805.4629.
- [11] K. Yokokawa, S. Sasaki, T. Hatsuda, and A. Hayashigaki, Phys.Rev. **D74**, 034504 (2006), hep-lat/0605009.
- [12] G.-Z. Meng et al. (CLQCD Collaboration), Phys.Rev. **D80**, 034503 (2009), 0905.0752.
- [13] L. Liu, H.-W. Lin, and K. Orginos, PoS **LATTICE2008**, 112 (2008), 0810.5412.
- [14] D. Mohler, S. Prelovsek, and R. Woloshyn (2012), 1208.4059.
- [15] L. Maiani and M. Testa, Phys.Lett. **B245**, 585 (1990).
- [16] M. Lüscher, Commun.Math.Phys. **105**, 153 (1986).
- [17] M. Lüscher, Nucl.Phys. **B354**, 531 (1991).
- [18] C. W. Bernard, T. Burch, K. Orginos, D. Toussaint, T. A. DeGrand, et al., Phys.Rev. **D64**, 054506 (2001), hep-lat/0104002.
- [19] M. G. Alford, W. Dimm, G. P. Lepage, G. Hockney, and P. B. Mackenzie, Phys. Lett. **B361**, 87 (1995), hep-lat/9507010.
- [20] K. Orginos, D. Toussaint, and R. L. Sugar (MILC), Phys. Rev. **D60**, 054503 (1999), hep-lat/9903032.
- [21] K. Orginos and D. Toussaint (MILC), Phys. Rev. **D59**, 014501 (1999), hep-lat/9805009.
- [22] D. Toussaint and K. Orginos (MILC), Nucl. Phys. Proc. Suppl. **73**, 909 (1999), hep-lat/9809148.
- [23] J. F. Lagae and D. K. Sinclair, Phys. Rev. **D59**, 014511 (1999), hep-lat/9806014.
- [24] G. P. Lepage, Phys. Rev. **D59**, 074502 (1999), hep-lat/9809157.
- [25] K. Orginos, R. Sugar, and D. Toussaint, Nucl. Phys. Proc. Suppl. **83**, 878 (2000), hep-lat/9909087.
- [26] S. Naik, Nucl. Phys. **B316**, 238 (1989).
- [27] Y. Shamir, Nucl. Phys. **B406**, 90 (1993), hep-lat/9303005.
- [28] V. Furman and Y. Shamir, Nucl. Phys. **B439**, 54 (1995), hep-lat/9405004.
- [29] D. B. Kaplan, Phys. Lett. **B288**, 342 (1992), hep-lat/9206013.
- [30] A. Hasenfratz and F. Knechtli, Phys. Rev. **D64**, 034504 (2001), hep-lat/0103029.
- [31] T. A. DeGrand, A. Hasenfratz, and T. G. Kovacs, Phys. Rev. **D67**, 054501 (2003), hep-lat/0211006.
- [32] T. A. DeGrand (MILC), Phys. Rev. **D69**, 014504 (2004), hep-lat/0309026.
- [33] S. Durr, C. Hoelbling, and U. Wenger, Phys. Rev. **D70**, 094502 (2004), hep-lat/0406027.
- [34] D. B. Renner et al. (LHP), Nucl. Phys. Proc. Suppl. **140**, 255 (2005), hep-lat/0409130.
- [35] R. G. Edwards et al. (LHPC), PoS **LAT2005**, 056 (2006), hep-lat/0509185.
- [36] A. Walker-Loud et al., Phys. Rev. **D79**, 054502 (2009), 0806.4549.
- [37] O. Bar, C. Bernard, G. Rupak, and N. Shores, Phys. Rev. **D72**, 054502 (2005), hep-lat/0503009.
- [38] B. C. Tiburzi, Phys. Rev. **D72**, 094501 (2005), hep-lat/0508019.

- [39] J.-W. Chen, D. O’Connell, and A. Walker-Loud, Phys. Rev. **D75**, 054501 (2007), hep-lat/0611003.
- [40] K. Orginos and A. Walker-Loud, Phys. Rev. **D77**, 094505 (2008), 0705.0572.
- [41] J.-W. Chen, D. O’Connell, and A. Walker-Loud, JHEP **04**, 090 (2009), 0706.0035.
- [42] J.-W. Chen, M. Golterman, D. O’Connell, and A. Walker-Loud, Phys. Rev. **D79**, 117502 (2009), 0905.2566.
- [43] W.-J. Lee and S. R. Sharpe, Phys. Rev. **D60**, 114503 (1999), hep-lat/9905023.
- [44] A. X. El-Khadra, A. S. Kronfeld, and P. B. Mackenzie, Phys. Rev. **D55**, 3933 (1997), hep-lat/9604004.
- [45] K. Symanzik, Nucl. Phys. **B226**, 187 (1983).
- [46] P. Chen, Phys. Rev. **D64**, 034509 (2001), hep-lat/0006019.
- [47] L. Liu, H.-W. Lin, K. Orginos, and A. Walker-Loud, Phys.Rev. **D81**, 094505 (2010), 0909.3294.
- [48] S. Beane, P. Bedaque, A. Parreno, and M. Savage, Phys.Lett. **B585**, 106 (2004), hep-lat/0312004.
- [49] F.-K. Guo, C. Hanhart, and U.-G. Meißner, Eur.Phys.J. **A40**, 171 (2009), 0901.1597.
- [50] Y.-R. Liu, X. Liu, and S.-L. Zhu, Phys.Rev. **D79**, 094026 (2009), 0904.1770.
- [51] L. Geng, N. Kaiser, J. Martin-Camalich, and W. Weise, Phys.Rev. **D82**, 054022 (2010), 1008.0383.
- [52] P. Wang and X. Wang (2012), 1204.5553.
- [53] J. Oller and E. Oset, Nucl.Phys. **A620**, 438 (1997), hep-ph/9702314.
- [54] J. Oller and E. Oset, Phys.Rev. **D60**, 074023 (1999), hep-ph/9809337.
- [55] J. Oller and U.-G. Meißner, Phys.Lett. **B500**, 263 (2001), hep-ph/0011146.
- [56] C. Hanhart, J. Pelaez, and G. Rios, Phys.Rev.Lett. **100**, 152001 (2008), 0801.2871.
- [57] M. Cleven, F.-K. Guo, C. Hanhart, and U.-G. Meißner, Eur.Phys.J. **A47**, 19 (2011), 1009.3804.
- [58] F. James and M. Roos, Comput.Phys.Comm. **10**, 343 (1975).
- [59] J. Beringer et al. (Particle Data Group), Phys. Rev. **D86**, 010001 (2012).
- [60] M. Frink and U.-G. Meißner, JHEP **0407**, 028 (2004), hep-lat/0404018.
- [61] J. M. Flynn and J. Nieves, Phys.Rev. **D75**, 074024 (2007), hep-ph/0703047.
- [62] T. Barnes, F. Close, and H. Lipkin, Phys.Rev. **D68**, 054006 (2003), hep-ph/0305025.
- [63] E. van Beveren and G. Rupp, Phys.Rev.Lett. **91**, 012003 (2003), hep-ph/0305035.
- [64] E. Kolomeitsev and M. Lutz, Phys.Lett. **B582**, 39 (2004), hep-ph/0307133.
- [65] F.-K. Guo, P.-N. Shen, H.-C. Chiang, R.-G. Ping, and B.-S. Zou, Phys.Lett. **B641**, 278 (2006), hep-ph/0603072.
- [66] D. Gamermann, E. Oset, D. Strottman, and M. Vicente Vacas, Phys.Rev. **D76**, 074016 (2007), hep-ph/0612179.
- [67] S. Weinberg, Phys.Rev. **137**, B672 (1965).
- [68] V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, and A. E. Kudryavtsev, Phys.Lett. **B586**, 53 (2004), hep-ph/0308129.
- [69] J. Laiho, E. Lunghi, and R. S. Van de Water, Phys.Rev. **D81**, 034503 (2010), 0910.2928.
- [70] <http://www.latticeaverages.org/>.
- [71] M. Golterman (2009), 0912.4042.

- [72] S. Sharpe (2006), hep-lat/0607016.
- [73] M. Della Morte and A. Jüttner, JHEP **1011**, 154 (2010), 1009.3783.
- [74] A. Jüttner, JHEP **1201**, 007 (2012), 1110.4859.